1 Topology, Knots and Knot Invariants

At the end of the 19th century, mathematicians discovered a new field of mathematics, called topology. Topology is similar to geometry but instead of studying properties related to measurements of objects, such as volume, length or size, it focuses on more fundamental properties. For example, more fundamental than an object's size is the number of components it is built out of.

A subfield within topology is knot theory. A mathematical knot is, roughly speaking, a looped piece of string in three-dimensional space. For example, the simplest kind of knot is a loop that is not knotted at all. In knot theory, we call this object the *unknot*. Looped strings may be twisted and knotted, however. Therefore, there are many different kinds of complicated knots. People sometimes perform magic tricks with knots: they show you a complicated looking knot, dramatically fumble around with it and suddenly it unknots nicely. Of course, this only works because the string was not knotted in the first place, but only appeared to be so. In mathematics, the same phenomenon leads to an interesting question: given any knot, can you determine whether its is actually not knotted at all, i.e. the unknot ?

To answer this question, mathematicians use a tool called *knot invariants*. One can think of knot invariants as labels that are attached to all knots. The content of the label can be any kind of mathematical object – an integer number, or a more complicated mathematical object, like a group[1]. We require only two properties of these labels:

1. they are defined for every knot one can possibly think of,

2. they are always the same label for the same knot, even if the knot appears in two forms that may look different.

To understand how knot invariants can help us answer our question, imagine some magical machine can determine a particular knot invariant for all knots. Given any knot, we pass it to the machine and verify whether the label assigned to this knot is the same as the label assigned to the unknot. Although knots do not have to be identical just because they are assigned the same label, the converse is true: If our knot is assigned a different label than the unknot, we know for certain that it cannot be the unknot.

2 Invariants from Quantum Groups

Unfortunately, for a long time, useful invariants were difficult to find in practice. In the 1980s, however, mathematicians found a method to produce a lot of new knot invariants. This method is based on quantum groups, very special instances of a class of mathematical objects called algebras. This is surprising: algebras are not topological objects, but belong to the field of algebra. Furthermore, there are a lot of quantum groups: for example, for every complex number, there are multiple possible quantum groups. A slogan to remember is that *every quantum group leads to a knot invariant*. It might thus seem like we have a lot of knot invariants and all our dreams have come true. Unfortunately, although we know that every quantum group yields a knot invariant in theory, actually determining the label of a particular invariant for a particular knot is challenging.

My research builds towards this problem. I am learning and thinking about different kinds of quantum groups, with the hope of eventually being able to explicitly calculate new knot invariants.

If you have some background in mathematics or are curious and want to learn more, here are some details: Recall that to any semisimple Lie algebra $\mathfrak{g}$ we can associate an associative $\mathbf{C}$-algebra $U(\mathfrak{g})$, called its universal enveloping algebra. Quantum groups are deformations of these universal enveloping algebras, along a deformation parameter $q \in \mathbf{C}$. That is, for any $q \in \mathbf{C} - \{0, \pm 1, \pm i\}$, we can define an associative algebra $U_q(\mathfrak{g})$ that abstracts properties of $U(\mathfrak{g})$. Furthermore, there exists a linear map $\Delta : U_q(\mathfrak{g}) \to U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ which behaves like an inverted multiplication. Algebras of this type are called bialgebras, or, if they have even more special properties, as is the case for quantum groups, *Hopf algebras*. Quantum groups are not the only examples of Hopf algebras, but they are the most interesting. An important result shows that we can build a knot invariant from every Hopf algebra, provided it is, in some sense, complicated enough. In particular, this is the case for essentially all quantum groups. Finally, quantum groups can be split into two classes: those for which the parameter q is a root of unity and those for which it is not. One way I hope to compute new invariants is by focussing on the former.

3 Applications and Societal Impact

Curious mathematicians are not the only people wondering about whether certain knots are unknots. In fact, some physical phenomena, such as DNA strings or, possibly, fundamental particles, can be modeled as mathematical knots. Scientists working in related field might therefore be able to classify or differentiate these objects using knot theory.

[1] Don't worry if this term does not mean anything to you